Journal homepage www.jzs.univsul.edu.iq
Journal of Zankoy Sulaimani
Part-A- (Pure and Applied Sciences)

The Nullity of Identifying Path Graph with Some Special Graphs

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Article info

Original: 16 July 2016
Revised: 5 November 2016
Accepted: 20 November 2016

Key Words: Path graph, identifying graphs, The nullity of a graph.

Abstract

In this paper the idea of high zero-sum weighting of new technique is used to find the nullity of some special graphs G_p, G_k, G_c which are constructed by identifying the path graph p_n of different order with star graph, complete graph, and cycle graph ,$p_n \bullet S_{1,m}$, $p_n \bullet K_p$ and $p_n \bullet c_m$, respectively. And it is proved that the nullity of these new graphs is increases or decreases according to the changing of identifying vertex in the path, or the other graphs, and with the order of path which is even or odd.

Introduction

For a graph G(V, E), let A(G) be the adjacency matrix of G. The graph G is said to be singular if A(G) is singular. The nullity of G is denoted by η(G), is the algebraic multiplicity of the zero eigenvalue in the spectra of G. In addition to its evident relevance in spectral graph theory, the nullity has a noteworthy applications for instance to networks in computer science and electrical circuits and chemistry. The recognition of this fact, it is starting point of an unprecedented activity in theoretical and mathematical chemistry and now a days referred to as chemical graph theory. [2], [7] and [12]. Most of the following concepts are found in [1, 3, 6, 8].

Let G be a graph with p vertices and q edges. The **degree** of vertex v in a graph G, denoted by $\deg_G(v)$, is the number of edges incident with v. A graph is **regular** if all vertices have the same degree. The **neighborhood** $N_G(v)$ of vertex v is the set of all vertices of G which are adjacent to v. Two edges incident with the same vertex are said to be adjacent edges.

Definition 1.1 [7]: The **adjacency matrix A(G)**=$[a_{i,j}]$ of labeled graph G with vertex set $V=\{v_1,v_2,\ldots,v_p\}$ is p×p matrix in which $a_{i,j}=1$ if v_i and v_j are adjacent and 0 if they are not.
The adjacent matrix of any graph G is always real symmetric (0,1) zero diagonal matrix.

Definition 1.2: The **characteristic polynomial** of the adjacency matrix A=A(G) of a graph G with p vertices denoted by $f(G,\lambda)$ with the convention that the coefficient of the highest order term is positive : $f(G,\lambda)$ $= Det(\lambda I_p - A)$ therefore, the characteristic polynomial of a graph G of order p is a polynomial of graph G with degree p: $f(G,\lambda)=\sum_{i=0}^{p} a_i \lambda^{p-i}$

Definition 1.3: The **eigenvalues** of a graph G of order p are defined to be the eigenvalues of the adjacency matrix associated with the graph G. That is if G has adjacency matrix A(G), then the eigenvalues of G are these p (not necessarily distinct) number λ which satisfies the determinant equation $Det(A-\lambda I_p)=0$

since the adjacent matrix A(G) is non-negative , real matrix, then all eigenvalues of A are real and always has a non-negative eigenvalue r such that all it's eigenvalues do not exceed r.

Equivalently, a number λ is an eigenvalue of G if there exists a non zero p×1 vector x (called an eigenvector of λ) such that AX=λX. [1] and [7]. The spectra $S_p(G)$ of G [8] is another matrix of two rows, the first of which contains the eigenvalues of G and the second the multiplicities of the eigenvalue.

Definition 1.4: [2] **Path graph** denoted by p_n is a graph with n vertices and n-1 edges, of order n and size n-1.

The **star graph $S_{1,m}$** is a complete bipartite graph $K_{1,m}$ of order m+1 and size m.

Definition 1.5: A cycle graph C_m $(m \geq 3)$ has m vertices and m edges; each vertex is of degree 2.

The nullity of a graph.

A graph G is said to be singular graph provided that its adjacency matrix A(G) is singular matrix. In the other words if and only if, zero is a root of the characteristic polynomial $f(G,\lambda)$.

Definition 2.1: Nullity of a graph

The nullity of a graph or the singularity of G denoted by η(G) is the multiplicity of zero as an eigenvalue in its spectra $S_p(G)$. [5]

Definition 2.2: A vertex **weighting** of a graph G is a function $f : V(G) \rightarrow R$ (the set of real number), that assign a real number to each vertex of G. The weighting of G is said to be non-trivial if there is at least one $v \in V(G)$ for which $f(v) \neq 0$.[16]

Definition 2.3 [5]: A non-trivial vertex weighting of a graph G is called a **zero-sum weighting** provided that for each $v \in V(G)$, $\sum f(w) = 0$ where the summation is taken over all $w \in N_G(v)$.

Out of all zero–sum weighting of a graph G, a **high zero-sum weighting** of G is one that uses maximum number of non- zero independent variables which denoted by $M_v(G)$.

Proposition 2.4 [15]**:** $M_v(G) = \eta(G)$.

Applying weighting techniques defined in 2.3 and proposition 2.4, some results are proved in [5,11, 13]. The problem is that how to find a relation between the graph structure and the nullity. This relation can perhaps be expressed by a set of rules by which nullity can be determined.

Theorem 2.5: The nullity of path p_n is equal to $\frac{(1-(-1)^n)}{2}$ means that p_n have nullity one if n is odd [9].

Theorem 2.6 [13]**:** The nullity of star graph $S_{1,m}$ is m-2 in general.

Theorem 2.7 [7]**:** The nullity of cycle graph C_m is one iff m≡0(mod4), i.e: C_m is singular iff m=4k for k=1,2,…,n.

Nullity of identifying path with other graphs.

In this paper, we investigate the problem of identifying the vertices of a path graph p_n such that we can uniquely identify any vertex in the path with the vertices of another graph G by examining the vertices that cover it.

Definition 3.1[8]**: vertex identifying graphs** $G_1 \bullet G_2$.

If G_1 and G_2 be two connected graphs with disjoint vertex sets V_1, V_2 , and u, v be any two vertices of G_1 and G_2 respectively, then the graph $G_1 \bullet G_2$ is obtained by identifying the vertices u and v as shown in Figure.1.

Example 3.2: Identifying vertex of two graphs G_1 and G_2 are shown in Figure .1.

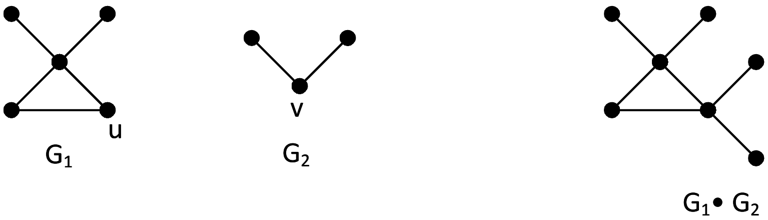


Fig .1. Graph $G_1 \bullet G_2$ obtained by identifying two vertices u and v

We construct a new graph (G_p) by identifying the different vertices of path p_n of different order n with some special graphs like star, cycle and complete, which causes the decreasing or increasing the nullity depending on the type of the graph and identifying vertex in p_n, if it is end vertices u_1,u_n neighbor of end vertices u_2,u_{n-1} or other vertices of degree two in p_n.

Let G_p be a graph obtained by identifying path p_n with star graph $S_{1,m}$, i.e: $G_p \cong P_n \bullet S_{1,m}$, then the nullity of G_p changes according to the change of n (the order of path, while n is odd or even) and the change of identifying vertex in the path and in star graph.

Example 3.3: The graph $G_p \cong P_5 \bullet S_{1,6}$. Shown in Figure .2, where the identifying vertices are u_3 in path and v_1 (one of end vertices) of star.

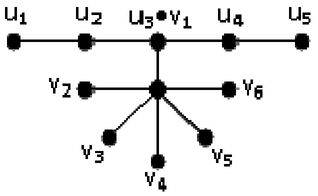


Fig.2. $P_5 \bullet S_{1,6}$

Theorem 3.4: For a path P_n (n is odd), the nullity of $G_p \cong P_n \bullet S_{1,m}$ is defined as follows:

i) $\eta(G_p) = m-1$ If identifying end vertex of star with any vertex of path $u_1,u_2,\ldots,u_n$.

ii) $\eta(G_p) = m+1$ If identifying center V_0 in star with even labels vertices of path, i.e: $u_2,u_4,\ldots,u_{n-1}$

iii) $\eta(G_p) = m-1$ If identifying center V_0 in star with odd labels vertices of path, i.e: $u_1,u_3,\ldots,u_n$

Proof :

Case (i): Let $f(v_i)$ be the weights of the each vertices v_i in G_p, such that the weights of the vertices $u_1,u_2,\ldots,u_n$ of path p_n, be of the form $-y_1,0,y_1,0,\ldots\ldots y_1$ (since the number of vertices is odd, then the weight of the last vertex is y_1 or $-y_1$) and the weights of end vertices $v_1,v_2,\ldots,v_m$ of star beginning from identifying vertex v_1 be $y_1,y_2,\ldots,y_m$ and the center of star v_0 weights zero as illustrated in Figure.3.

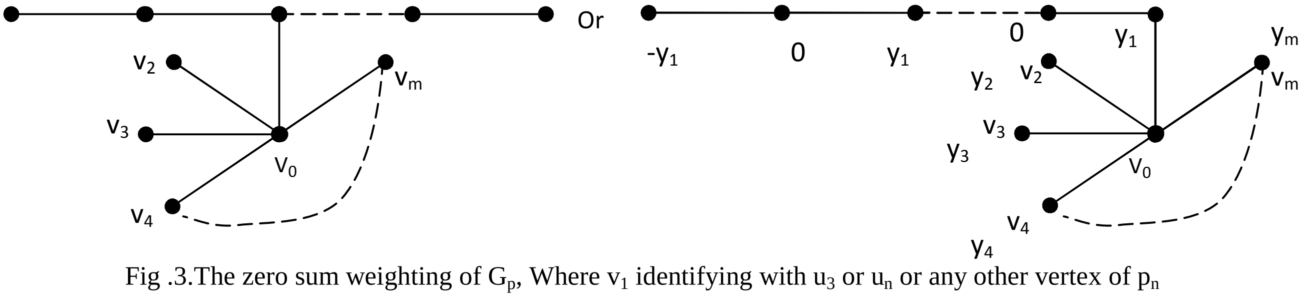


Fig .3.The zero sum weighting of G_p, Where v_1 identifying with u_3 or u_n or any other vertex of p_n

To apply the condition $\sum_{w\in N(v_i)} f(w)=0$ for each vertex in G_p, we get $y_1+y_2+\ldots+y_{m-1}+y_m=0$ put $y_m=-y_1-y_2-\ldots-y_{m-1}$,then the high zero sum weighting for the graph G_p exists, which uses exactly (m-1) independent variables. Therefore $\eta(G_p)=m-1$.

Where if the identifying vertex in path is in even order u_2, u_4, ….., u_{n-1} which has 0 weights, the non-zero independent variables used to get the zero sum weighting $\sum_{w\in N(v_i)} f(w)=0$,for all vertex v in G_p in this case is:

$-y_1+y_1+0+y_2+y_3+\ldots\ldots+y_m=0$, (zero is the weight of identifying vertices $u_{2h}\bullet v_1$)
Means that $y_m=y_1-y_1-y_2-y_3-\ldots\ldots-y_{m-1}$. Which is used exactly m-1 independent variables.

Case (ii): The weights of vertices of the path in G_p is in form $x_1,0,x_2,0,-x_2,0,\ldots,x_2$ and the weight of vertices of the star graph $S_{1,m}$ is $f(v_i)=y_i$, i=1,2,…,m and $f(v_0)=0$ as shown in Figure .4.

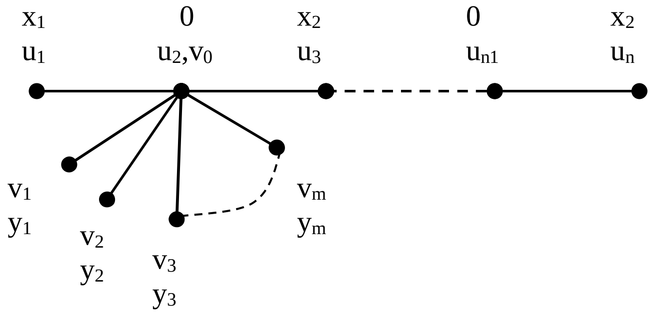


Fig.4. The high zero sum weighting for v_0 with even vertex of p_n

To get $\sum_{w\in N(v_i)} f(w)=0$, we must have $x_1+x_2+\sum_{i=1}^{m} y_i=0$ implies that $x_2=-x_1-\sum_{i=1}^{m} y_i$, to get the zero-sum weighting, then a high zero sum weighting for G_p exists, and uses exactly (m+1) non-zero independent variables, that is the nullity of G_p is m+1.

Case (iii):
Let the weights of the vertices of the path for this case is shown in Figure 5 be of the form x_1, $0,-x_1$, ... , $0,x_1$ or $-x_1$ where the weighting of odd order vertices $f(v_i)=x$ or –x for i=1,3,5,…,n.

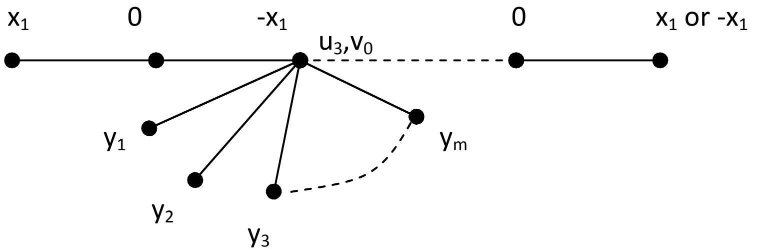


Fig .5. The zero-sum weighting of G_p if identifying v_0 with odd order vertex of p_n

For applying the zero sum weighting we have $\sum_{i=1}^{m} y_i=0$, to get $\sum_{w\in N(v_i)} f(w)=0$, then $y_m=-y_1-y_2-\ldots.-y_{m-1}$ and $f(v_3)=f(v_0)=-x_1$ where u_3 or v_0 are the neighbors of end vertices of star graph $S_{1,m}$,so it must be zero to get the zero-sum weighting. Then the only non-zero independent variable used to get zero sum weighting of G_p in this case are $y_1,y_2,y_3,\ldots,y_{m-1}$ which is the high zero sum weighting then $\eta(G_p) = m-1$.

Theorem 3.5: If n is even in path, the nullity of G_p is defined as follow:

i) $\eta(G_p) = m-1$ If identifying end vertices of the star with any vertex of the path.

ii) $\eta(G_p) = m$ if identifying vertex V_0 (center vertex of the star) in a star with any vertex of P_n.

Proof: If n is even, then P_n is nonsingular; Theorem 2.5 means that $M_v(P_n) = 0$ and it does not possess the zero sum weighting.

i) If identifying any end vertex of star with vertices of P_n as shown in Figure .6, the weights of the vertices of path must be zero; $f(u_i) = 0$, i =1,2,…,n and $f(v_\circ) = 0$, since $v_\circ$ is the only neighbor of m vertices in the star. But the weights of end vertices of star be of the form $y_1,y_2,\ldots,y_m$ beginning with identifying vertex v_1 with u_i, then $y_1=0$ and to apply $\sum_{w\in N(v_i)} f(w) = 0$,

$y_2+y_3+\ldots+y_m=0 \rightarrow y_m= -y_2-y_3-\ldots-y_{m-1}$, then the high zero sum weighting $M_v(G)$ Exist for the graph (G_p) which used exactly m-2 non zero independent variables.

$\therefore M(G_p) = m-2$.

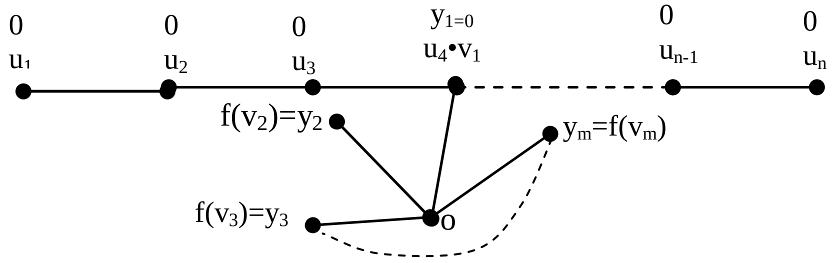


Fig.6. The zero-sum weighting of u4•v1 in Gp (end vertex of star with any vertex in path)

ii) Two cases are considered:

a) If the identifying vertices in path have the odd subscript $u_1,u_3,\ldots,u_{n-1}$ the weighting of identifying vertex u_i and u_{i-1} is zero, $f(u_i)=f(u_{i+1})=0$ and the weights of the other vertices of path in G_p , $f(u_1),f(u_2),\ldots,f(u_{i-2})$ and $f(u_{i+1}),f(u_{i+2}),\ldots,f(u_{n-1})$ is of the form x,0,-x,… , as shown in Figure.7.

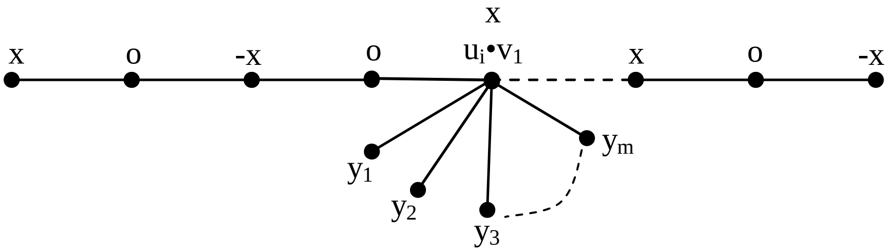


Fig .7. Identifying vertices ui where i is odd

To apply $\sum_{w\in n(v_i)} f(w)=0$ for each vertex v_i in G_p, $x+y_1+y_2+\ldots+y_m=0 \rightarrow x=-y_1-y_2-\ldots-y_m$, then the graph G_p, possess the zero sum and uses exactly m non-zero independent variables in high zero sum weighting then $M(G_p)=m$.

b) But if identifying vertex is even vertices in path $f(u_i)=f(u_{i+1})=0$, and the weights of the other vertices must be of the form x,o,-x,…., then $\eta(G_p)=m$ as indicated in Figure.8.

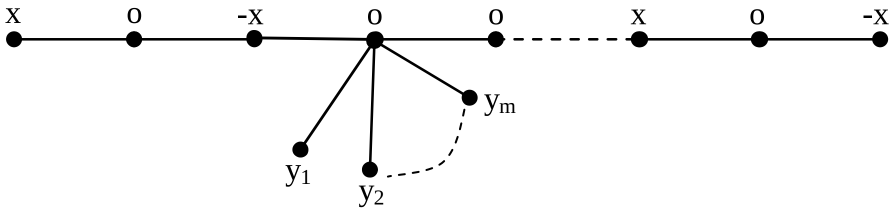


Fig .8. Identifying vertices u_i where i is even

Remark: All the eigenvalues λ_i, $i=1,2,...,n-\eta(G_p)$ (all eigenvalues rather than zero eigenvalues, i.e ±a, ±b,…) of the new graph G_p is symmetric around zeros eigenvalue.

Now we investigate a new class of graphs by identifying operation, by construct a graph G_k by identifying path P_n with complete graph K_p.

Definition 3.6: Complete graph denoted by K_p, is a graph of order p in which each vertex is adjacent to all other vertices and have degree p-1, and its non-singular graph for each i, i=2,…,p [6].

Let $G_k \cong P_n \bullet K_p$, determined the nullity of this graph where the identifying vertex is either end vertices or others (vertices of degree 2) in the path, with any vertex of K_p(K_p is a regular graph and all vertices have the same degree) in the following theorems:

Theorem 3.7: For a path P_n (n is odd), the nullity of graph G_k is defined as follow:

i) $\eta(G_k)=1$ If identifying any vertex of K_p with vertices (u_2 or u_{n-1}) of path P_n.

ii) $\eta(G_k)=0$ Otherwise.

Proof:

i): Since complete graph is non-singular, means that it is not possess the zero sum weighting and any variables used for zero sum weighting must be zero and the non-zero independent variables used us weights of path P_n in the graph G_k is only one as shown in Figure 9, then the high zero sum weighting exist and equal to one which is means that the nullity of G_k is one.

ii): If identifying any other vertices in path neither u_2 nor $u_{n-1}$the weights of the identifying vertex is x or –x to apply $\sum_{w\in N(v_i)} f(w)=0$, for all v_i in G_k, then x must be zero since its neighborhood set are vertices of K_p, all with weights zero. So $\eta(G_k)=0$.

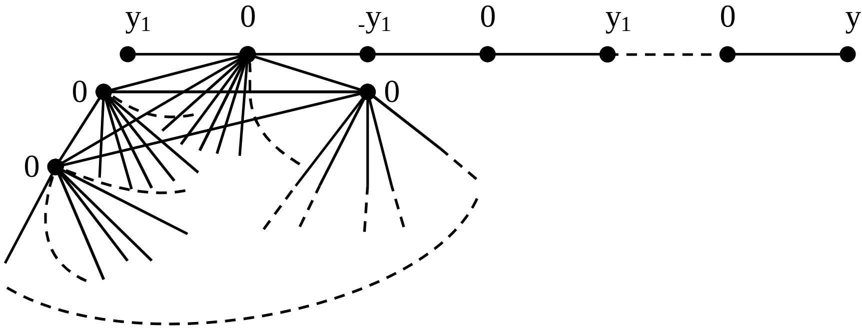


Fig .9. The zero sum weighting for p_n with k_p

Theorem.3.8: For a path Pn (n is even), the nullity is defined to be $\eta(G_k) = 0$ for any vertex in path Pn.

Proof: It is obvious since no path nor any complete graph are singular, if we used any variables to get any zero-sum weighting, each variable must be zero.

4. The nullity of identifying path with cycle:

In graph theory, the term cycle may refer to a closed path. If repeated vertices are allowed it is more often called a closed walk, in this paper we take a simple cycle with no repeated vertices nor edges other than the ending vertices.

Now the new graph in this section G_c defined to be the graph obtained by identifying different vertices u_i of the path with a vertex in cycle C_m of a different order.

We obtain a new graph which is different from the basic graphs in order, size and degree of vertices and especially in nullity. Let $G_c \cong P_n \bullet C_m$, we define the nullity of this new graph in the following theorems:

Theorem 4.1: For a path Pn (n is odd), the nullity of G_c is defined as:

Case 1: if m is odd in cycle graph

i) $\eta(G_c) = 1$ if identifying any vertex of cycle C_m with the even subscript vertices of path Pn.

ii) $\eta(G_c) = 0$ if identifying any vertex of cycle C_m with the odd subscript vertices of path Pn.

Case 2: if m is even in cycle graph Cm.

i) if $m \equiv 0(\text{mod}\,4)$ in cycle c_m, then $\eta(G_c) = 2$

ii) if $m \neq 0(\text{mod}\,4)$, then

$\eta(G_c) = 2$ If identifying vertex is even in path P_n

$\eta(G_c) = 0$ if identifying vertex is odd in path P_n

Proof:

Case 1: i), The weights of vertices of path, where identifying vertex is even must be of the form x,o,-x,…,x,o,-x as shown in figure (10). That is the weights of even subscript order vertices is always zero, and $f(v_i)=0, \forall v_i \in V(c_m)$, then the zero-sum weighting $\sum_{w_i \in N(v_i)} f(w_i) = 0$ exist and used only one non-zero variable x, means that the nullity is one.

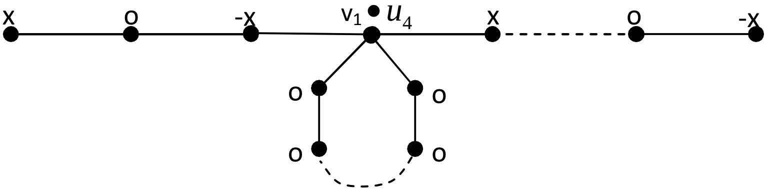


Fig .10. The identifying vertex has even subscript in path P_n

(ii) Identifying odd subscript vertices Pn as shown in Figure 11.
$f(u_i)=x$, or $-x$, $i=1,3,\ldots,n$ and to get the zero sum weighting we have $x+(-x)=0$, since for identifying vertex v_i(i is odd) to apply $\sum_{w_i\in N(v_i)} f(w_i)=0$, $0 + 0 + -x + 0 + x + \ldots\ldots=0$ means that ,there is no any variable used to get the zero sum weighting of this graph. So the nullity of this graph is zero, $\eta(G_c)=o$.

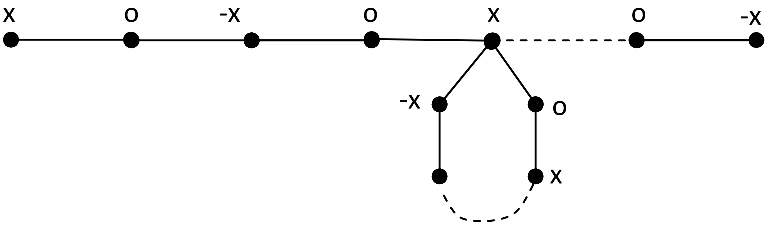
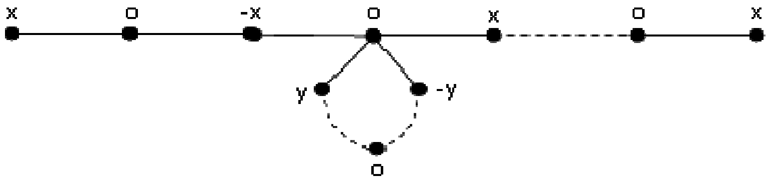


Fig .11. The identifying vertex has odd labels in P_n where n is odd

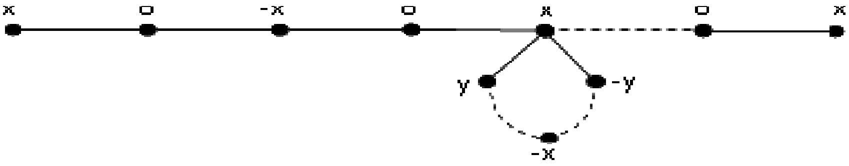
case 2:

i) If m≡ 0(mod 4):

The weights of identifying vertex in path part is zero if it is even and x or –x if it is odd so depending on the weights of identifying vertex $f(u_i)$, i=1,…,n , changes the weights of vertices of cycle as illustrated in Figure 12, 13 to be o, -y, o, y,…. for first case and x, -y, -x, y, ….. for second case, in two cases the zero-sum weighting exist and uses exactly two independent variables x and y implies that the high zero-sum weighting exist and $M_v(G)=2$
then $\eta(G_c)=2$.



f Cm.



ex of Cm.

ii) If $m \neq 0(\mathrm{mod}4)$:

If identifying vertex in path is even, i.e 2,4,…. , n , the weights of the vertices of path is x, 0, -x,0,…. , f(ui)=0 if i is even.
The weights of the vertices of the cycle must be of the form y,o,-y,… and the weights of even vertices of u_i is zero so we use exactly x,y two independent variables to get the zero sum weighting as shown in Figure.14.

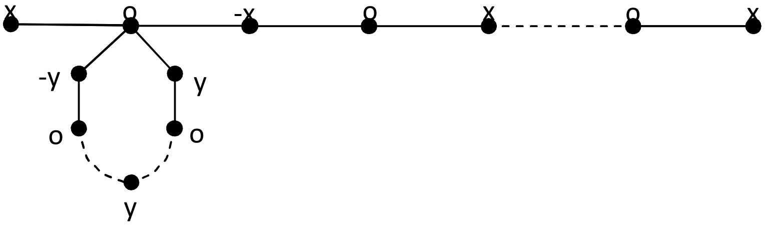


Fig. 14 The identifying vertex has even labels in Pn when n is odd

But if identifying vertex is odd, then f(u$_i$)=x or –x for i=1,3,5,…,n-1 which don't possess the zero sum weighting since x=y=0, so $\eta(c_m)=0$ for this case, see Figure.15.

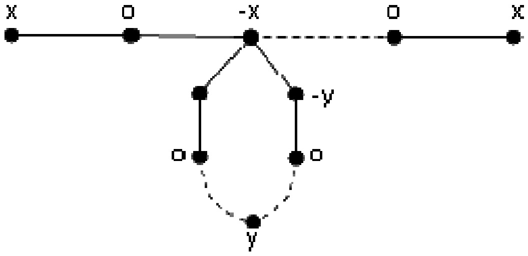


Fig.15. Not posses zero sum weighting

Theorem 4.2: For a path Pn (n is even), the nullity of $G_c \cong P_n \bullet C_m$ is defined as follows:

i) $\eta(G_c)=1$ if m is even

ii) $\eta(G_c)=0$ if m is odd

Proof:

i): To apply $\sum_{w_i \in N(v_i)} f(w_i)=0$ for each vertex of new graph G$_c$ as shown in Figure 16, then,

$f(u_i)=0, \forall i=1,\dots,n$ u$_i$ the vertices of path part and f(v$_i$)=0,x,0,-x,0,…, v$_i$ the vertices of cycle part.

Beginning from identifying vertex, then in the zero sum weighting we use only x (one variable), to get the high zero sum weighting of the graph G$_c$, means that $\eta(G_c)=1$.

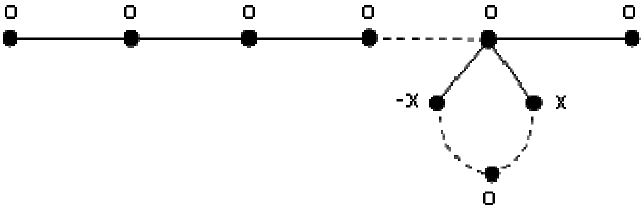


Fig.16.

Fig.16 The identifying vertex of Pn where n is even with vertex of Cm, (m is even)

ii): If m is odd, it is obvious, the graph does not possess the zero-sum weighting, then $\eta(G_c)=0$.

Conclusion: In general the nullity of special graphs are defined, in this work we define some new graphs which constructed by the identifying operation between the path graph and others, and we find the nullity of these graphs by simple way.

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